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Radon problems for hyperboloids

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Задачи Радона для гиперболоидов

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Abstract. We offer a variant of Radon transforms for a pair $\mathcal{X}$ and $\mathcal{Y}$ of hyperboloids in $\mathbb{R}^3$ defined by $[x,x]=1$ and $[y,y]=-1, y_1\geqslant 1$, respectively, here $[x,y]=-x_1y_1+x_2y_2+x_3y_3$. For a kernel of these transforms we take $\delta([x,y])$, $\delta(t)$ being the Dirac delta function. We obtain two Radon transforms $\mathcal{D}(\mathcal{X})\to\mathcal{C}^\infty(\mathcal{Y})$ and $\mathcal{D}(\mathcal{Y})\to\mathcal{C}^\infty(\mathcal{X})$. We describe kernels and images of these transforms. For that we decompose a sesqui-linear form with the kernel $\delta([x,y])$ into inner products of Fourier components.

Keywords: hyperboloids; Radon transform; distributions; representations; Poisson and Fourier transforms

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Аннотация. Мы предлагаем некоторый вариант преобразований Радона для пары $\mathcal{X}$ и $\mathcal{Y}$ гиперболоидов в $\mathbb{R}^3$, определенных уравнениями $[x,x]=1$ and $[y,y]=-1, y_1\geqslant 1$, соответственно, здесь $[x,y]=-x_1y_1+x_2y_2+x_3y_3$. В качестве ядра этих преобразований мы берем $\delta([x,y])$, где $\delta(t)$ – дельта-функция Дирака. Мы получаем два преобразования Радона $\mathcal{D}(\mathcal{X})\to\mathcal{C}^\infty(\mathcal{Y})$ и $\mathcal{D}(\mathcal{Y})\to\mathcal{C}^\infty(\mathcal{X})$. Мы описываем ядра и образы этих преобразований. Для этого мы разлагаем полуторалинейную форму с ядром $\delta([x,y])$ по скалярным произведениям компонент Фурье.

Ключевые слова: гиперболоиды; преобразование Радона; обобщенные функции; представления; преобразования Пуассона и Фурье

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In this paper we offer a variant of Radon transforms for a pair of dual hyperboloids in $\mathbb{R}^3$: the one-sheeted hyperboloid $\mathcal{X}$: $[x,x]=1$ ($[x,y]=-x_1y_1+x_2y_2+x_3y_3$) and the upper sheet of the two-sheeted hyperboloid $\mathcal{Y}$: $[y,y]=-1, y_1\geqslant 1$ (the Lobachevsky plane). For a kernel of these transforms we take $\delta([x,y])$, $x\in\mathcal{X}$, $y\in\mathcal{Y}$, $\delta(t)$ being the Dirac delta function. This kernel gives two Radon transforms $R:\mathcal{D}(\mathcal{X})\to\mathcal{C}^\infty(\mathcal{Y})$ and $R^*:\mathcal{D}(\mathcal{Y})\to\mathcal{C}^\infty(\mathcal{X})$. We describe kernels and images of these transforms. For that we consider a sesqui-linear form with the kernel $\delta([x,y])$ and write the decomposition of this form into inner products of Fourier components. Results of this paper were announced in [4].

1. Hyperboloids

Let G be the group $\mathrm{SO}_0(1,2)$, it is a connected group of linear transformations of $\mathbb{R}^3$, preserving the form

$$[x,y]=-x_1y_1+x_2y_2+x_3y_3.$$

We consider that G acts on $\mathbb{R}^3$ from the right. In accordance with this we write vectors in the row form.

Let us take the following basis of the Lie algebra $\mathfrak{g}$ of the group G:

$$L_0=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ L_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ L_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}. \tag{1.1}$$

The Casimir element in the universal enveloping algebra of $\mathfrak{g}$ is $(1/2)\Delta_{\mathfrak{g}}$, where

$$\Delta_{\mathfrak{g}}=-L_0^2+L_1^2+L_2^2. \tag{1.2}$$

Consider subgroups K, H, A of the group G generating by elements L_0, L_1, L_2, respectively. The subgroup K is a maximal compact subgroup of G.

Denote by $\mathcal{X}$ the one-sheeted hyperboloid $[x,x]=1$, and by $\mathcal{Y}$ the upper sheet of the two-sheeted hyperboloid $[y,y]=-1$, $y_1\geqslant 1$ (we consider that the x_1-axis goes up). These hyperboloids $\mathcal{X}$ and $\mathcal{Y}$ are homogeneous spaces of the group G with respect to translations $x\mapsto xg$, namely, $\mathcal{X}=G/H$ and $\mathcal{Y}=G/K$. The subgroups H and K are stabilizers of points $x^0=(0,0,1)\in\mathcal{X}$ and $y^0=(1,0,0)\in\mathcal{Y}$ respectively.

These hyperboloids have a G-invariant metric. It gives rise to the measures dx and dy and the Laplace–Beltrami operators $\Delta_{\mathcal{X}}$ and $\Delta_{\mathcal{Y}}$ respectively (all are G-invariant).

As local coordinates on the hyperboloids we can take any two variables from x_1,x_2,x_3. For $\mathcal{Y}$ it is natural to take y_2,y_3. Then we have

$$\begin{aligned}
dx &= |x_1|^{-1}dx_2dx_3,\ \ dy=y_1^{-1}dy_2dy_3,\\
\Delta_{\mathcal{X}} &= \frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}+D_1^2+D_1,\ \ D_1=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2},\\
\Delta_{\mathcal{Y}} &= \frac{\partial^2}{\partial y_2^2}-\frac{\partial^2}{\partial 3_2^2}+D_1^2+D_1,\ \ D_1=y_2\frac{\partial}{\partial y_2}+y_3\frac{\partial}{\partial y_3}.
\end{aligned}$$

If M is a manifold, then $\mathcal{D}(M)$ denotes the space of compactly supported infinitely differentiable $\mathbb{C}$-valued functions on M, with the usual topology, and $\mathcal{D}'(M)$ denotes the space of distributions on M — of antilinear continuous functionals on $\mathcal{D}(M)$. For a differentiable representation of a Lie group, we retain the same symbol for the corresponding representations of its Lie algebra and of the universal enveloping algebra.

Let us denote by $U_{\mathcal{X}}$ and $U_{\mathcal{Y}}$ representations of our group G by translations on functions on $\mathcal{X}$ and $\mathcal{X}$ respectively (quasiregular representations):

$$(U_{\mathcal{X}}(g)f)(x) = f(xg),\ \ (U_{\mathcal{X}}(g)f)(y) = f(yg).$$

The representations $U_{\mathcal{X}}$ and $U_{\mathcal{Y}}$ on the spaces $L^2(\mathcal{X}, dx)$ and $L^2(\mathcal{Y}, dy)$ are unitary with respect to the inner products

$$\langle F, f\rangle_{\mathcal{X}} = \int_{\mathcal{X}} F(x)\overline{f(x)}dx,\ \ \langle F, f\rangle_{\mathcal{Y}} = \int_{\mathcal{Y}} F(y)\overline{f(y)}dy. \tag{1.3}$$

We have

$$U_{\mathcal{X}}(\Delta_{\mathfrak{g}}) = \Delta_{\mathcal{X}},\ U_{\mathcal{Y}}(\Delta_{\mathfrak{g}}) = \Delta_{\mathcal{Y}}. \tag{1.4}$$

2. Representations of the group $\mathrm{SO}_0(1,2)$

Recall some material about the principal non-unitary series of representations of the group $G = \mathrm{SO}_0(1,2)$, see, for example, [5]. Let $\mathcal{C}^+$ be the cone $[x,x] = 0$, $x_1 > 0$. The group G acts transitively on it. For $\sigma \in \mathcal{C}$, let $\mathcal{D}_\sigma(\mathcal{C}^+)$ be the space of C^∞ functions φ on $\mathcal{C}^+$ homogeneous of degree σ:

$$\varphi(tx) = t^\sigma \varphi(x),\ t > 0.$$

Let T_σ be the representation of G acting on this space by translations:

$$(T_\sigma(g)\varphi)(x) = \varphi(xg).$$

Take the section S of the cone $\mathcal{C}^+$ by the plane $x_1 = 1$, it is a circle consisting of points $s = (1, \sin\alpha, \cos\alpha)$. The Euclidean measure on S is $ds = d\alpha$. For a function φ on S, sometimes we write $\varphi(\alpha)$ instead of $\varphi(s)$. The representation T_σ can be realized on the space $\mathcal{D}(S)$ as follows (index 1 indicates the first coordinate of a vector):

$$(T_\sigma(g)\varphi)(s) = \varphi\left(\frac{sg}{(sg)_1}\right)(sg)_1^\sigma.$$

The element $\Delta_{\mathfrak{g}}$, see (1.2), goes to a scalar operator:

$$T_\sigma(\Delta_{\mathfrak{g}}) = \sigma(\sigma+1)E. \tag{2.1}$$

The Hermitian form

$$\langle \psi, \varphi\rangle_S = \int_S \psi(s)\overline{\varphi(s)}ds \tag{2.2}$$

is invariant with respect to the pair $(T_\sigma, T_{-\overline{\sigma}-1})$, i. e.

$$\langle T_\sigma(g)\psi, \varphi\rangle_S = \langle \psi, T_{-\overline{\sigma}-1}(g^{-1})\varphi\rangle_S. \tag{2.3}$$

This formula follows from $d\widetilde{s} = (sg)_1^{-1}\, ds$, where $\widetilde{s} = (sg)/(sg)_1$.

Define an operator A_σ in $\mathcal{D}(S)$:

$$(A_\sigma\varphi)(s) = \int_S (-[s,u])^{-\sigma-1}\varphi(u)du.$$

The integral converges absolutely for $\mathrm{Re}\sigma < -1/2$ and can be continued as a meromorphic function to the whole σ-plane. It has simple poles at $\sigma \in -1/2+\mathbb{N}$. Here and further $\mathbb{N} = \{0,1,2,...\}$. For A_σ we have

$$\langle A_\sigma\psi, \varphi\rangle_S = \langle \psi, A_{\overline{\sigma}}\varphi\rangle_S. \tag{2.4}$$

The operator A_σ intertwines T_σ and $T_{-\sigma-1}$, i. e.

$$T_{-\sigma-1}(g)A_\sigma = A_\sigma T_\sigma(g),\ g \in G.$$

A sesqui-linear form $\langle A_\sigma\psi, \varphi\rangle_S$ is invariant with respect to the pair $(T_\sigma, T_{\overline{\sigma}})$. In particular, for $\sigma \in \mathbb{R}$, this form is an invariant Hermitian form for T_σ.

Take a basis $\psi_m(\alpha) = \mathrm{e}^{im\alpha}$, $m \in \mathbb{Z}$, in $\mathcal{D}(S)$. It consists of eigenfunctions of A_σ:

$$A_\sigma\psi_m = a(\sigma,m)\psi_m, \tag{2.5}$$

where

$$a(\sigma,m) = 2^{\sigma+2}\pi(-1)^m\frac{\Gamma(-2\sigma-1)}{\Gamma(-\sigma+m)\Gamma(-\sigma-m)}. \tag{2.6}$$

The composition $A_\sigma A_{-\sigma-1}$ is a scalar operator:

$$A_\sigma A_{-\sigma-1} = \frac{1}{8\pi\omega(\sigma)}\cdot E$$

where $\omega(\sigma)$ is a "Plancherel measure" (see (5.2)):

$$\omega(\sigma) = \frac{1}{32\pi^2}(2\sigma+1)\cot\sigma\pi. \tag{2.7}$$

The representation T_σ can be extended to the space $\mathcal{D}'(S)$ of distributions on S by formula (2.3) where ψ is a distribution and $\langle\psi,\varphi\rangle_S$ is the value of the distribution ψ at a test function φ. It is an extension in fact, since $\mathcal{D}(S)$ can be embedded into $\mathcal{D}'(S)$ by means of the form (2.2), namely, we assign to a function $\psi \in \mathcal{D}(S)$ the functional $\varphi \mapsto \langle\psi,\varphi\rangle_S$ in $\mathcal{D}'(S)$.

Similarly the operator A_σ can be extended to the space $\mathcal{D}'(S)$ by means of formula (2.4).

If σ is not integer, then T_σ is irreducible and T_σ is equivalent to $T_{-\sigma-1}$ (by A_σ or $\widehat{A}_\sigma$).

Let $\sigma \in \mathbb{Z}$, $n \in \mathbb{N}$. Subspaces $V_{\sigma,+}$ and $V_{\sigma,+}$ spanned by ψ_m for which $m \geqslant -\sigma$ and $m \leqslant \sigma$ respectively are invariant. For $\sigma < 0$ they are irreducible and orthogonal to each other. For $\sigma \geqslant 0$ their intersection E_σ is irreducible and has dimension $2\sigma+1$.

Let $V_n^d = \mathcal{D}(S)/E_n$ and $V_{-n-1}^d = V_{-n-1,+} + V_{-n-1,-}$. Let us denote by T_σ^d, $\sigma \in \mathbb{Z}$, the representation on V_σ^d generated by T_σ. The operator A_n vanishes on E_n and gives rise to the equivalence $T_n^d \sim T_{-n-1}^d$.

There are four series of unitarizable irreducible representations: the *continuous series* consisting of representations T_σ with $\sigma=-1/2+i\rho,\ \rho\in\mathbb{R}$, the inner product is (2.2); the *complementary series* consisting of T_σ with $-1<\sigma<1$, the inner product is $\langle A_\sigma\psi,\varphi\rangle_S$ with a factor; the *holomorphic* and *antiholomorphic* series. We need only their sum T_σ^d. We shall call T_σ^d the representations of *discrete series.* For $\varphi\in\mathcal{D}(S)$, denote by $\widetilde{\varphi}$ the coset of φ modulo E_n. Then the invariant inner product $(\cdot\,,\cdot)_n$ for T_n^d is

$$(\widetilde{\psi},\widetilde{\varphi})_n=c_n\langle A_n\psi,\varphi\rangle_S,\quad c_n=a(n,n+1)^{-1}. \tag{2.8}$$

3. Poisson and Fourier transforms

First we determine distributions θ in $\mathcal{D}'(S)$ invariant with respect to the subgroup H under the representations T_σ. We shall use the following notation for a character of the group $\mathbb{R}^*$:

$$t^{\lambda,m}=|t|^\lambda(\operatorname{sgn}t)^m,$$

where $t\in\mathbb{R}^*=\mathbb{R}\backslash\{0\},\ \lambda\in\mathbb{C},\ m\in\mathbb{Z}$. In fact this character depends only on m modulo 2. Here and further the sign "$\equiv$" means the congruence modulo 2.

It is easy to check that the distribution

$$\theta_{\sigma,\varepsilon}=s_3^{\sigma,\varepsilon}=[x^0,s]^{\sigma,\varepsilon},$$

where $\sigma\in\mathbb{C},\ \varepsilon=0,1,$ is H-invariant. Sometimes we write an integer instead of ε with the same parity as ε. As a function of σ, $\theta_{\sigma,\varepsilon}$ is a meromorphic function — with simple poles at points $\sigma\in-1-\varepsilon-2\mathbb{N}$. Its residue at $\sigma=-n-1,\ n\equiv\varepsilon,$ is the distribution $\mathrm{const}\cdot\delta^{(n)}(s_3)$ concentrated at two points $s=(1,\pm1,0)$. Here $\delta(t)$ is the Dirac delta function on the real line (a linear continuous functional on $\mathcal{D}(\mathbb{R})$). The space of H-invariants has dimension 2 for $\sigma\neq-n-1,\ n\in\mathbb{N},$ and dimension 3 for $\sigma=-n-1$. Every irreducible subfactor for $T_\sigma,\ \sigma\in\mathbb{Z},$ contains, up to a factor, precisely one H-invariant. In particular, $\theta_{-n-1,n+1}$ and $\theta_{n,n+1}$ have non-zero projections into $V'_{-n-1,\pm}$ and $\mathcal{D}'(S)/V'_{n,\mp}$ respectively.

The operator A_σ carries $\theta_{\sigma,\varepsilon}$ to $\theta_{-\sigma-1,\varepsilon}$ with a factor:

$$A_\sigma\theta_{\sigma,\varepsilon}=j(\sigma,\varepsilon)\theta_{-\sigma-1,\varepsilon}, \tag{3.1}$$

where

$$j(\sigma,\varepsilon)=2^{-\sigma}\pi^{-1/2}\Gamma\left(-\sigma-\frac{1}{2}\right)\Gamma(\sigma+1)\left[1-(-1)^\varepsilon\cos\sigma\pi\right]. \tag{3.2}$$

It is easy to check that

$$j(\sigma,\varepsilon)j(-\sigma-1,\varepsilon)=(8\pi\omega(\sigma))^{-1},$$

where $\omega(\sigma)$ is given by (2.7) or (5.2). The factor $j(\sigma,\varepsilon)$ has simple poles at $\sigma\in-1/2+\mathbb{N}$.

By a general scheme [3], the H-invariant $\theta_{\sigma,\varepsilon}$ gives rise to the Poisson kernel $P_{\sigma,\varepsilon}(x,s)=[x,s]^{\sigma,\varepsilon},\ x\in\mathcal{X},\ s\in S$. This kernel gives rise to two transforms. The first of them, the *Poisson transform* $P_{\sigma,\varepsilon}:\mathcal{D}(S)\to C^\infty(\mathcal{X})$ is a linear continuous operator defined as follows:

$$(P_{\sigma,\varepsilon}\varphi)(x)=\int_S[x,s]^{\sigma,\varepsilon}\varphi(s)ds.$$

It intertwines $T_{-\sigma-1}$ with $U_{\mathcal{X}}$, therefore, its image consists of eigenfunctions of the Laplace–Beltrami operator:

$$\Delta_{\mathcal{X}} \circ P_{\sigma,\varepsilon} = \sigma(\sigma+1)P_{\sigma,\varepsilon}$$

of parity ε (see (2.1) and (1.4)). As a function in σ, the Poisson transform behaves like $\theta_{\sigma,\varepsilon}$: it depends on σ meromorphically and has simple poles at $\sigma \in -1-\varepsilon-2\mathbb{N}$.

Formula (3.1) gives

$$P_{\sigma,\varepsilon}A_{\sigma} = j(\sigma,\varepsilon)P_{-\sigma-1,\varepsilon}. \tag{3.3}$$

Consider $\sigma \in \mathbb{Z}$. The transform $P_{-n-1,n+1}$ vanishes on E_n, it generates an operator on $\mathcal{D}(S)/E_n$ which intertwines T_n^d with $U_{\mathcal{X}}$. The Poisson transform $P_{n,n+1}$ considered on V_{-n-1}^d intertwines T_{-n-1}^d with $U_{\mathcal{X}}$. By (3.3), $P_{n,n+1}$ has the same image as $P_{-n-1,n+1}$.

The second transform generated by the Poisson kernel is the *Fourier transform* $F_{\sigma,\varepsilon}$: $\mathcal{D}(\mathcal{X}) \to \mathcal{D}(S)$ defined by

$$\left(F_{\sigma,\varepsilon}f\right)(s) = \int\limits_{\mathcal{X}} [x,s]^{\sigma,\varepsilon} f(x)dx.$$

It is meromorphic in σ with simple poles at points $\sigma \in -1-\varepsilon-2\mathbb{N}$. It intertwines $U_{\mathcal{X}}$ with T_{σ}. It follows from (3.1) that

$$A_{\sigma}F_{\sigma,\varepsilon} = j(\sigma,\varepsilon)F_{-\sigma-1,\varepsilon}. \tag{3.4}$$

For a function $f \in \mathcal{D}(\mathcal{X})$, let us call two functions $F_{\sigma,\varepsilon}f$, $\varepsilon = 0,1$, the *Fourier components* of f corresponding to the representation T_{σ}.

The Fourier and Poisson transforms are conjugate to each other with respect to forms (1.3) and (2.2):

$$\langle P_{\sigma,\varepsilon}\varphi, f\rangle_{\mathcal{X}} = \langle \varphi, F_{\overline{\sigma},\varepsilon}f\rangle_S.$$

This relation allows to extend the Poisson transform to distributions on S.

Consider the reducible case. The *Fourier transform* F_n corresponding to T_n^d is defined as the map of $\mathcal{D}(\mathcal{X})$ to $\mathcal{D}(S)/E_n$ which assigns to $f \in \mathcal{D}(\mathcal{X})$ the corresponding coset of the function $F_{n,n+1}f$. By (2.8) and (3.4) we have

$$(F_nf, F_nh)_n = d_n\langle F_{-n-1,n+1}f, F_{n,n+1}h\rangle_S, \quad d_n = 2n!^2/\pi(2n+1)!.$$

The *Fourier transform* corresponding to T_{-n-1}^d is $F_{-n-1,n+1}$.

The representation T_{σ} has one up to a factor K-invariant, it is the function τ_{σ} equal to 1 identically on S:

$$\tau_{\sigma}(s) = [y^0,s]^{\sigma} = 1.$$

The representations of the discrete series have no K-invariants.

The corresponding Poisson transform Q_{σ} : $\mathcal{D}(S) \to C^{\infty}(\mathcal{Y})$ and Fourier transform $\mathcal{D}(\mathcal{Y}) \to \mathcal{D}(S)$ are defined by

$$\begin{aligned}
(Q_{\sigma}\varphi)(y) &= \int_S [-y,s]^{\sigma}\varphi(s)ds, \\
\left(G_{\sigma}h\right)(s) &= \int_{\mathcal{Y}} [-y,s]^{\sigma}h(y)dy.
\end{aligned}$$

Notice that $[-y,s]>0$ for all $y\in\mathcal{Y}$ and $s\in S$.

The Poisson transform Q_σ intertwines $T_{-\sigma-1}$ with $U_{\mathcal{Y}}$, therefore, its image consists of eigenfunctions of the Laplace–Beltrami operator:

$$\Delta_{\mathcal{Y}}\circ Q_{\sigma,\varepsilon}=\sigma(\sigma+1)Q_{\sigma,\varepsilon}. \tag{3.5}$$

4. Spherical functions

Let $\sigma\in\mathbb{C}$, $\varepsilon=0,1$. Let us define a spherical function $\Psi_{\sigma,\varepsilon}$ on the hyperboloid $\mathcal{Y}$ as follows

$$\Psi_{\sigma,\varepsilon}(y) = \langle T_\sigma(g)\theta_{\sigma,\varepsilon},\tau_{-\overline{\sigma}-1}\rangle_S \tag{4.1}$$
$$= \langle \theta_{\sigma,\varepsilon},T_{-\overline{\sigma}-1}(g^{-1})\tau_{-\overline{\sigma}-1}\rangle_S$$
$$= \int_S \theta_{\sigma,\varepsilon}[-y,s]^{-\sigma-1}ds, \tag{4.2}$$

where $g\in G$ is such that $y^0g=y$. As the distribution $\theta_{\sigma,\varepsilon}$ does, the spherical function $\Psi_{\sigma,\varepsilon}$ is given by an integral absolutely convergent for $\mathrm{Re}\sigma>-1$ and can be continued analytically in σ to a meromorphic function. It has poles where $\theta_{\sigma,\varepsilon}$ has and of the same (the first) order.

The function $\Psi_{\sigma,\varepsilon}(y)$ is a function of class C^∞ on $\mathcal{Y}$ invariant with respect to H:

$$\Psi_{\sigma,\varepsilon}(yh)=\Psi_{\sigma,\varepsilon}(y),\ \ h\in H.$$

Therefore, it depends on $y_3=[x^0,y]$ only:

$$\Psi_{\sigma,\varepsilon}(y)=\Phi_{\sigma,\varepsilon}(y_3), \tag{4.3}$$

where $\Phi_{\sigma,\varepsilon}(c)$ is a function in $C^\infty(\mathbb{R})$.

Lemma 4.1. *The function $\Phi_{\sigma,\varepsilon}$ has the following integral representation:*

$$\Phi_{\sigma,\varepsilon}(c)=\int_0^{2\pi}\left(c+\sqrt{c^2+1}\cdot\cos\alpha\right)^{\sigma,\varepsilon}d\alpha. \tag{4.4}$$

P r o o f. Let us take in (4.1) as g the matrix $a=\exp t\,L_2$, see (1.1), in A:

$$a=\begin{pmatrix}\cosh t & 0 & \sinh t\\ 0 & 1 & 0\\ \sinh t & 0 & \cosh t\end{pmatrix}.$$

We have

$$\left(T_\sigma(a)\,\theta_{\sigma,\varepsilon}\right)(s)=[x^0,sa]^{\sigma,\varepsilon}=(\sinh t+s_3\cosh t)^{\sigma,\varepsilon}.$$

By (4.1), the value of $\Psi_{\sigma,\varepsilon}$ at the point $y^0a=(\cosh t,0,\sinh t)$ is just (4.4) with $c=\sinh t$. □

It follows from (4.4) that the function $\Phi_{\sigma,\varepsilon}$ has parity ε:

$$\Phi_{\sigma,\varepsilon}(-c)=(-1)^\varepsilon\Phi_{\sigma,\varepsilon}(c).$$

Equality (4.2) shows that the spherical function $\Psi_{\sigma,\varepsilon}$ is the Poisson transform of the H-invariant:

$$\Psi_{\sigma,\varepsilon} = Q_{-\sigma-1}\theta_{\sigma,\varepsilon}. \tag{4.5}$$

Consider $\Psi_{\sigma,\varepsilon}$ as a distribution on $\mathcal{Y}$:

$$\langle \Psi_{\sigma,\varepsilon}, f\rangle_{\mathcal{Y}} = \int_{\mathcal{Y}} \Psi_{\sigma,\varepsilon}(y)\overline{f(y)}dy, \tag{4.6}$$

where $f \in \mathcal{D}(\mathcal{Y})$. The right hand side in (4.6) can be rewritten as an iterated integral, then we obtain:

$$\langle \Psi_{\sigma,\varepsilon}, f\rangle_{\mathcal{Y}} = \int_{-\infty}^{\infty} \Phi_{\sigma,\varepsilon}(c)\overline{(Mf)(c)}dc, \tag{4.7}$$

where

$$(Mf)(c) = \int_{\mathcal{Y}} \delta(y_3 - c) f(y)dy,$$

The map M assigns to a function f its integrals over H-orbits. It is a continuous operator from $\mathcal{D}(\mathcal{Y})$ onto $\mathcal{D}(\mathbb{R})$.

Lemma 4.2. *The value* (4.6) *is expressed in terms of Fourier components:*

$$\langle \Psi_{\sigma,\varepsilon}, f\rangle_{\mathcal{Y}} = \langle \theta_{\sigma,\varepsilon}, G_{-\sigma-1} f\rangle_S. \tag{4.8}$$

P r o o f. Let $h(y)$ be a majorant of the function $f(y)$, depending on y_1 only. Then for $\mathrm{Re}\sigma > -1$ the right hand side in (4.8) is majorized by the integral

$$\int_0^{2\pi} |\cos\alpha|^\tau d\alpha \int_{\mathcal{Y}} \Big|[y,s]\Big|^{-\tau-1} h(y)dy, \tag{4.9}$$

where $\tau = \mathrm{Re}\sigma$. In fact, the integral over $\mathcal{Y}$ here does not depend on s. Therefore, integral (4.9) converges absolutely and the order of integration can be inverted. So we get equality (4.8) for $\mathrm{Re}\sigma > -1$. To other σ this equality is extended by analycity. □

Let Φ be a distribution on $\mathcal{Y}$ invariant with respect to H. Assign to it two things: a *convolution* with Φ of functions f in $\mathcal{D}(\mathcal{Y})$ and a sesqui-linear functional K on the pair ($\mathcal{D}(\mathcal{X})$, $\mathcal{D}(\mathcal{Y})$). The convolution $\Phi \star f$ is the following function on $\mathcal{X}$:

$$\begin{aligned}(\Phi \star f)(x) &= \langle \Phi, U_{\mathcal{Y}}(g)\overline{f}\rangle_{\mathcal{Y}}\\ &= \int_{\mathcal{Y}} \Phi(y) f(yg)dy,\end{aligned}$$

the functional is:

$$\begin{aligned}K(\Phi|h,f) &= \langle h, \overline{\Phi} \star f\rangle_{\mathcal{X}}\\ &= \int_{\mathcal{X}} h(x)\langle \Phi, U_{\mathcal{Y}}(g) f\rangle_{\mathcal{Y}}\, dx\\ &= \int_{\mathcal{X}\times\mathcal{Y}} \Phi(yg^{-1})h(x)\overline{f(y)}dx\, dy,\end{aligned}$$

where $h \in \mathcal{D}(\mathcal{X})$, $f \in \mathcal{D}(\mathcal{Y})$ and g is an arbitrary element in G carrying x^0 to x. Since Φ is H-invariant, these formulae do not depend on the choice of g for given x. The convolution is a linear map $\mathcal{D}(\mathcal{Y}) \to C^\infty(\mathcal{X})$, intertwining $U_\mathcal{Y}$ and $U_\mathcal{X}$:

$$\Phi \star (U_\mathcal{Y}(g) f) = U_\mathcal{X}(g)\, (\Phi \star f)\,.$$

For the spherical function $\Psi_{\sigma,\varepsilon}$, the convolution and the functional are expressed in terms of the Poisson and Fourier transforms:

$$\begin{aligned} (\Psi_{\sigma,\varepsilon} \star f)\,(x) &= (P_{\sigma,\varepsilon} G_{-\sigma-1} f)\,(x), \\ K(\Psi_{\sigma,\varepsilon} | h, f) &= \langle F_{\sigma,\varepsilon} h, G_{-\overline{\sigma}-1} f \rangle_S. \end{aligned} \tag{4.10}$$

The kernel $K_{\sigma,\varepsilon}(x,y)$ of the functional (4.10) is

$$K_{\sigma,\varepsilon}(x,y) = \int_S [x,s]^{\sigma,\varepsilon} [-y,s]^{-\sigma-1} ds.$$

Lemma 4.3. *The function $\Psi_{\sigma,\varepsilon}$ has the following property of symmetry in σ:*

$$\Psi_{-\sigma-1,\varepsilon} = -\frac{1 + (-1)^\varepsilon \cos \sigma\pi}{\sin \sigma\pi} \cdot \Psi_{\sigma,\varepsilon}. \tag{4.11}$$

P r o o f. By Lemma 4.2, (3.1), (2.4), (2.5) and Lemma 4.2 again we have:

$$\begin{aligned} \langle \Psi_{-\sigma-1,\varepsilon}, f \rangle_\mathcal{Y} &= \langle \theta_{-\sigma-1,\varepsilon}, G_{\overline{\sigma}} f \rangle_S \\ &= j(\sigma,\varepsilon)^{-1} \langle A_\sigma \theta_{\sigma,\varepsilon}, G_{\overline{\sigma}} f \rangle_S \\ &= j(\sigma,\varepsilon)^{-1} \langle \theta_{\sigma,\varepsilon}, A_{\overline{\sigma}} G_{\overline{\sigma}} f \rangle_S \\ &= a(\sigma,0) j(\sigma,\varepsilon)^{-1} \langle \theta_{\sigma,\varepsilon}, G_{-\overline{\sigma}-1} f \rangle_S \\ &= a(\sigma,0) j(\sigma,\varepsilon)^{-1} \langle \Psi_{\sigma,\varepsilon}, f \rangle_\mathcal{Y}. \end{aligned}$$

Substituting here values of $a(\sigma,0)$ and $j(\sigma,\varepsilon)$ from (2.6) and (3.2), we get (4.11). □

Lemma 4.4. *The spherical function $\Psi_{\sigma,\varepsilon}$ is an eigenfunction of the Laplace–Beltrami operator:*

$$\Delta_\mathcal{Y} \Psi_{\sigma,\varepsilon} = \sigma(\sigma+1) \Psi_{\sigma,\varepsilon}. \tag{4.12}$$

P r o o f. The function $\Psi_{\sigma,\varepsilon}$ is the Poisson transform of the function $\theta_{\sigma,\varepsilon}$, see (4.5). It remains to remember (3.5). □

On functions depending on $y_3 = c$ only, the operator $\Delta_\mathcal{Y}$ becomes to the following differential operator (the H-radial part of $\Delta_\mathcal{Y}$):

$$L = (c^2+1) \frac{\partial^2}{\partial c^2} + 2c \frac{\partial}{\partial c}. \tag{4.13}$$

Lemma 4.5. *The function $\Phi_{\sigma,\varepsilon}$, see (4.3) and (4.4), is an eigenfunction of L:*

$$L \Phi_{\sigma,\varepsilon} = \sigma(\sigma+1) \Phi_{\sigma,\varepsilon}.$$

The lemma follows immediately from Lemma 4.4.

Theorem 4.1. *The spherical function* $\Psi_{\sigma,\varepsilon}(y)$ *is expressed in terms of the Legendre functions* P_σ *(see [2, Ch. III]) of the imaginary argument:*

$$\Psi_{\sigma,\varepsilon}(y) = \frac{2\pi}{e^{i\sigma\pi/2} + (-1)^\varepsilon e^{-i\sigma\pi/2}} \Big\{ P_\sigma(iy_3) + (-1)^\varepsilon P_\sigma(-iy_3) \Big\}. \tag{4.14}$$

P r o o f. Denote for brevity:

$$P_\sigma(it) = B_\sigma(t), \tag{4.15}$$

also for a function $\varphi(t)$ on $\mathbb{R}$ we shall denote

$$\widehat{\varphi}(t) = \varphi(-t).$$

Equality (4.14) is equivalent to the following expression of the function $\Phi_{\sigma,\varepsilon}$:

$$\Phi_{\sigma,\varepsilon}(c) = \frac{2\pi}{e^{i\sigma\pi/2} + (-1)^\varepsilon e^{-i\sigma\pi/2}} \Big\{ B_\sigma(c) + (-1)^\varepsilon \widehat{B}_\sigma(c) \Big\}. \tag{4.16}$$

So we have to prove (4.16).

The Legendre function $P_\sigma(z)$ is analytic in the z-plane with the cut $(-\infty, -1]$, satisfies the equation:

$$\left((z^2-1)\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z} \right) w = \sigma(\sigma+1) w \tag{4.17}$$

and has the integral representation

$$P_\sigma(z) = \frac{1}{2\pi} \int_0^{2\pi} \left(z + \sqrt{z^2-1}\cos\alpha \right)^\sigma d\alpha. \tag{4.18}$$

Let σ be not integer. Then the functions $P_\sigma(z)$ and $\widehat{P}_\sigma(z)$ form a basis of solutions of equation (4.17). For $z = ic$ equation (4.17) becomes the equation:

$$Lw = \sigma(\sigma+1)w.$$

In virtue of Lemma 4.5 the function $\Phi_{\sigma,\varepsilon}$ is a linear combination of functions B_σ and $\widehat{B}_\sigma$. Coefficients of this linear combination could be found out by computing values of functions $\Phi_{\sigma,\varepsilon}$, B_σ and $\widehat{B}_\sigma$ and their derivatives at the point $c = 0$, using (4.6) and explicit expressions [2, 3.4(20),(23)]. But it is more convenient for us to find them in another way.

Let z tend to ic, $c \in \mathbb{R}$, in (4.18) such that $\operatorname{Re} z > 0$. We get:

$$B_\sigma(c) = \frac{1}{2\pi} e^{i\sigma\pi/2} \int_0^{2\pi} \left(c + \sqrt{c^2+1}\cos\alpha - i0 \right)^\sigma d\alpha. \tag{4.19}$$

Denote

$$Z_\sigma(c) = \int_0^{2\pi} \left(c + \sqrt{c^2+1}\cos\alpha - i0 \right)_+^\sigma d\alpha.$$

Then

$$\widehat{Z}_\sigma(c) = \int_0^{2\pi} \left(c + \sqrt{c^2+1}\cos\alpha - i0 \right)_-^\sigma d\alpha.$$

Applying to (4.19) the formula:

$$(t-i0)^\sigma = t_+^\sigma + \mathrm{e}^{-i\sigma\pi} t_-^\sigma,$$

we obtain

$$B_\sigma = \frac{1}{2\pi}\Big[\mathrm{e}^{i\sigma\pi/2} Z_\sigma + \mathrm{e}^{-i\sigma\pi/2}\widehat{Z}_\sigma\Big], \tag{4.20}$$

whence

$$\widehat{B}_\sigma = \frac{1}{2\pi}\Big[\mathrm{e}^{-i\sigma\pi/2} Z_\sigma + \mathrm{e}^{i\sigma\pi/2}\widehat{Z}_\sigma\Big]. \tag{4.21}$$

From (4.20) and (4.21) we have

$$Z_\sigma = \frac{\pi}{i\sin\sigma\pi}\Big[\mathrm{e}^{i\sigma\pi/2} B_\sigma - \mathrm{e}^{-i\sigma\pi/2}\widehat{B}_\sigma\Big], \tag{4.22}$$

$$\widehat{Z}_\sigma = \frac{\pi}{i\sin\sigma\pi}\Big[-\mathrm{e}^{-i\sigma\pi/2} B_\sigma + \mathrm{e}^{i\sigma\pi/2}\widehat{B}_\sigma\Big]. \tag{4.23}$$

Since

$$\Phi_{\sigma,\varepsilon} = Z_\sigma + (-1)^\varepsilon \widehat{Z}_\sigma.$$

we obtain (4.16) by (4.22) and (4.22). □

Let us establish some estimates for spherical functions of the continuous series ($\sigma=-1/2+i\rho$). They show that values of these spherical functions at f decrease rapidly when their parameter ρ tends to infinity.

Theorem 4.2. *Let $\sigma=-1/2+i\rho$, $\rho\in\mathbb{R}$. For any compact set $W\subset\mathcal{Y}$, there exists a number $C>0$ such that for any $f\in\mathcal{D}(\mathcal{Y})$ with the support in W the following estimate holds:*

$$|\langle\Psi_{\sigma,\varepsilon}, f\rangle_{\mathcal{Y}}| \leqslant C\cdot\max_y\big|\big(\Delta_{\mathcal{Y}}^m f\big)(y)\big|\,(\rho^2+1/4)^{-m},\ \ m\in\mathbb{N}. \tag{4.24}$$

P r o o f. Take $h\in\mathcal{D}(\mathcal{Y})$ depending on y_1 only, such that $h(y)\geqslant 0$, $h(y)=1$ on W. Then μh, where $\mu=\max|f(y)|$, is a majorant for f depending on y_1 only. Arguing as in the proof of Lemma 4.2, we obtain

$$|\langle\Psi_{\sigma,\varepsilon}, f\rangle_{\mathcal{Y}}| \leqslant C\mu, \tag{4.25}$$

where C is the number

$$C = \int_0^{2\pi} |\cos\alpha|^{-1/2} d\alpha \int_{\mathcal{Y}} [-y,s]^{-1/2} h(y)\,dy.$$

Now apply the estimate (4.25) to the function $\Delta_{\mathcal{Y}}^m f$, $m\in\mathbb{N}$, transfer the operator $\Delta_{\mathcal{Y}}$ to the function $\Psi_{\sigma,\varepsilon}$, since it is self-adjoint, and use (4.12). Since $|\sigma(\sigma+1)|=\rho^2+1/4$ for $\sigma=-1/2+i\rho$, we get (4.24). □

Let us write expressions of $\Psi_{\sigma,\varepsilon}$ for σ integer. In the notation $\Psi_{\sigma,\varepsilon}$ sometimes it is convenient to write an integer instead of ε with the same parity as ε.

Let $n \in \mathbb{N}$. Let first $\sigma = n$. For $\Psi_{n,n+1}$ we have to evaluate an indeterminacy in (4.14). We have

$$\begin{aligned} \Psi_{n,n}(y) &= 2\pi i^{-n} P_n(iy_3), \\ \Psi_{n,n+1} &= -4i^{1-n} Q_n^*(iy_3), \end{aligned}$$

where $P_n(z)$ is the Legendre polynomial and $Q_n^*(z)$ is the Legendre function which differs from the Legendre function of the second kind $Q_n(z)$ by the cut on the z-plane: for $Q_n(z)$ one takes the cut $[-1,1]$, but for $Q_n^*(z)$ one has to take the cut $(-\infty,-1]\cup[1,\infty)$; therefore, we have:

$$Q_n^*(z) = \frac{1}{2} P_n(z) \ln \frac{1+z}{1-z} - W_{n-1}(z),$$

cf. [2, 3.6(24)], where the principal branch of the logarithm is taken and $W_{n-1}(z)$ is a polynomial of degree $n-1$ indicated in [2, 3.6.2].

For $\sigma = -n-1$ we use the relation (4.11). For $\varepsilon \equiv n$ the function $\Psi_{\sigma,\varepsilon}$ has a pole at $\sigma = -n-1$ because of $\theta_{\sigma,\varepsilon}$. We have

$$\begin{aligned} \Psi_{-n-1,n+1} &= 0, \\ \mathrm{Res}_{\sigma=-n-1}\, \Psi_{\sigma,n} &= (-1)^{n+1}\,(2/\pi)\,\Psi_{n,n}. \end{aligned}$$

5. Eigenfunction decomposition of the radial part of the Laplace-Beltrami operator

In this Section we obtain the eigenfunction decomposition of the operator (see (4.13))

$$L = (c^2+1)\frac{\partial^2}{\partial c^2} + 2c\frac{\partial}{\partial c}$$

defined on the real line $\mathbb{R}$. We use the function $\Phi_{\sigma,\varepsilon}(c)$, see (4.3) and (4.4). Recall that it has parity ε and satisfies the equation:

$$Lw = \sigma(\sigma+1)w.$$

Let us denote by (φ,ψ) the $L^2(\mathbb{R})$ inner product of functions φ,ψ:

$$(\varphi,\psi) = \int_{-\infty}^{\infty} \varphi(c)\overline{\psi(c)}\,dc.$$

Theorem 5.1. *There is the following eigenfunction decomposition of the operator L:*

$$(\varphi,\psi) = \int_{-\infty}^{\infty} \omega(\sigma) \sum_{\varepsilon} (\varphi,\Phi_{\sigma,\varepsilon})(\Phi_{\sigma,\varepsilon},\psi)\Big|_{\sigma=-1/2+i\rho} d\rho, \tag{5.1}$$

where

$$\omega(\sigma) = \frac{1}{32\pi^2}(2\sigma+1)\cot\sigma\pi \tag{5.2}$$

so that

$$\omega\left(-\frac{1}{2}+i\rho\right) = \frac{1}{16\pi^2}\rho\tanh\rho\pi.$$

P r o o f. Let us write the resolvent $R_\lambda = (\lambda E - L)^{-1}$ of the operator L. Let $h \in L^2(\mathbb{R})$ and $R_\lambda h = f$, then $h = (\lambda E - L)f$, so that

$$Lf - \lambda f = -h. \tag{5.3}$$

Let us take λ in the form $\lambda = \sigma(\sigma+1)$. The correspondence $\sigma \mapsto \lambda$ maps the half plane $\mathrm{Re}\sigma > -1/2$ onto the λ-plane with the cut $(-\infty, -1/4]$ one-to-one.

Let f_1, f_2 be eigenfunctions of the operator L with the eigenvalue $\lambda = \sigma(\sigma+1)$ with $\mathrm{Re}\sigma > -1/2$. They behave at infinity ($\pm\infty$) as $A|c|^\sigma + B|c|^{-\sigma-1}$. Let us take them such that they are square integrable at $+\infty$ and $-\infty$ respectively. Then for $c \to +\infty$:

$$\begin{aligned} f_1(c) &\sim \quad B_1 c^{-\sigma-1}, \\ f_2(c) &\sim A_2 c^\sigma + B_2 c^{-\sigma-1}, \end{aligned}$$

and for $c \to -\infty$:

$$\begin{aligned} f_1(c) &\sim C_1 |c|^\sigma + D_1 |c|^{-\sigma-1}, \\ f_2(c) &\sim \quad D_2 |c|^{-\sigma-1}. \end{aligned}$$

The wronskian W of these functions is

$$W = \frac{W_0}{c^2+1}, \quad W_0 = (2c+1) B_1 A_2.$$

We have already several eigenfunctions: $P_\sigma(ic)$, $P_\sigma(-ic)$, $Z_\sigma(c)$, $\widehat{Z}_\sigma(c)$, $\Phi_{\sigma,\varepsilon}(c)$, $\varepsilon = 0, 1$. By [2, 3.2(18)] the Legendre functions behave when $c \to +\infty$ as follows:

$$\begin{aligned} P_\sigma(ic) &\sim p(\sigma) \cdot \mathrm{e}^{i\sigma\pi/2} \cdot c^\sigma + p(-\sigma-1) \cdot \mathrm{e}^{i(-\sigma-1)\pi/2} \cdot c^{-\sigma-1}, \\ P_\sigma(-ic) &\sim p(\sigma) \cdot \mathrm{e}^{-i\sigma\pi/2} \cdot c^\sigma + p(-\sigma-1) \cdot \mathrm{e}^{i(\sigma+1)\pi/2} \cdot c^{-\sigma-1}, \end{aligned}$$

where

$$p(\sigma) = 2^\sigma \pi^{-1} \mathrm{B}\left(\sigma + \frac{1}{2}, \frac{1}{2}\right),$$

$\mathrm{B}(a,b)$ being the Euler beta function.

By (4.22), (4.22) it gives that when $c \to +\infty$ we have

$$\begin{aligned} Z_\sigma(c) &\sim 2\pi \cdot p(\sigma) \cdot c^\sigma - \frac{2\pi}{\sin\sigma\pi} \cdot p(-\sigma-1) \cdot c^{-\sigma-1}, \\ \widehat{Z}_\sigma(c) &\sim 2\pi \cdot \cot\sigma\pi \cdot p(-\sigma-1) \cdot c^{-\sigma-1}. \end{aligned}$$

Therefore, as a mentioned-above basis f_1, f_2 of solutions of the equation $Lw = \lambda w$, $\lambda = \sigma(\sigma+1)$, we can take the pair $\widehat{Z}_\sigma$, Z_σ. Then

$$\begin{aligned} W_0 &= (2\sigma+1) \cdot 2\pi p(\sigma) \cdot 2\pi \cot\sigma\pi \cdot p(-\sigma-1) \\ &= 4\pi. \end{aligned}$$

Therefore, the solution f of equation (5.3) is

$$f(c) = \frac{1}{4\pi}\Big\{\widehat{Z}_\sigma(c) \int_{-\infty}^c Z_\sigma(t) h(t) dt + Z_\sigma(c) \int_c^\infty \widehat{Z}_\sigma(t) h(t) dt\Big\}.$$

Thus, for $\mathrm{Im}\lambda \neq 0$, the resolvent R_λ is an integral operator with the kernel

$$K_\lambda(c,t) = \begin{cases} (1/4\pi)\widehat{Z}_\sigma(c)Z_\sigma(t), \ c > t, \\ (1/4\pi)Z_\sigma(c)\widehat{Z}_\sigma(t), \ c < t, \end{cases} \tag{5.4}$$

here $\lambda = \sigma(\sigma+1)$ and σ belongs to the half plane $\mathrm{Re}\sigma > -1/2$ with the cut along the real axis.

Let $\varphi, \psi \in L^2(\mathbb{R})$. By the Titchmarsh–Kodaira theorem [1, XIII] we have

$$(\varphi,\psi) = \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty} (R_{\lambda-i\varepsilon}\varphi,\psi)\, d\lambda - \int_{-\infty}^{\infty} (R_{\lambda+i\varepsilon}\varphi,\psi)\, d\lambda\right].$$

Let us pass to σ. Then $d\lambda = (2\sigma+1)d\sigma$ and we denote $S_\sigma = R_\lambda$. The operator function S_σ is analytic in the half plane $\mathrm{Re}\sigma > -1/2$. Therefore,

$$(\varphi,\psi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (2\sigma+1)(S_\sigma\varphi,\psi)\Big|_{\sigma=-1/2+i\rho} d\rho.$$

We can keep here only the even part in ρ of the integrand:

$$(\varphi,\psi) = \frac{1}{4\pi}\int_{-\infty}^{\infty} (2\sigma+1)\left((S_\sigma - S_{-\sigma-1})\varphi,\psi\right)\Big|_{\sigma=-1/2+i\rho} d\rho.$$

Let us compute the kernel $M_\sigma(c,t)$ of the operator $S_\sigma - S_{-\sigma-1}$. Let $c > t$. By (5.4) we have

$$M_\sigma(c,t) = \frac{1}{4\pi}\Big\{\widehat{Z}_\sigma(c)Z_\sigma(t) - \widehat{Z}_{-\sigma-1}(c)Z_{-\sigma-1}(t)\Big\}$$

Let us insert here (4.22) and (4.22) and use that the Legendre function P_σ is unchanged under $\sigma \mapsto -\sigma-1$. We obtain (recall notation (4.15)):

$$M_\sigma(c,t) = -\frac{\pi\cos\sigma\pi}{2\sin^2\sigma\pi}\Big\{B_\sigma(c)\widehat{B}_\sigma(t) + \widehat{B}_\sigma(c)B_\sigma(t)\Big\}. \tag{5.5}$$

For $c < t$, we obtain the same expression.

Further, if $\sigma = -1/2 + i\rho$, then for the Legendre function P_σ on the imaginary axis we have

$$\overline{P_\sigma(ic)} = P_{\overline{\sigma}}(-ic) = P_{-\sigma-1}(-ic) = P_\sigma(-ic),$$

or, in terms of B_σ:

$$\overline{B_\sigma(c)} = \widehat{B}_{\overline{\sigma}}(c) = \widehat{B}_{-\sigma-1}(c) = \widehat{B}_\sigma(c).$$

Therefore, equality (5.5) gives

$$\begin{aligned}(\varphi,\psi) \ &= \ -\int_{-\infty}^{\infty} \frac{(2\sigma+1)\cos\sigma\pi}{8\sin^2\sigma\pi}\Big\{(\varphi,B_\sigma)(B_\sigma,\psi) \\ &+ \ (\varphi,\widehat{B}_\sigma)(\widehat{B}_\sigma,\psi)\Big\}\Big|_{\sigma=-1/2+i\rho} d\rho.\end{aligned} \tag{5.6}$$

This formula is the desired eigenfunction decomposition – in the basis B_σ, $\widehat{B}_\sigma$.

Now let us pass in (5.6) from B_σ, $\widehat{B}_\sigma$ to $\Phi_{\sigma,\varepsilon}$, $\varepsilon = 0,1$, by

$$\begin{aligned} B_\sigma \ &= \ \frac{1}{2\pi}\left(\cos\frac{\sigma\pi}{2}\cdot\Phi_{\sigma,0} + i\sin\frac{\sigma\pi}{2}\cdot\Phi_{\sigma,1}\right), \\ \widehat{B}_\sigma \ &= \ \frac{1}{2\pi}\left(\cos\frac{\sigma\pi}{2}\cdot\Phi_{\sigma,0} + i\sin\frac{\sigma\pi}{2}\cdot\Phi_{\sigma,1}\right),\end{aligned}$$

then we obtain (5.1). □

6. Decomposition of a sesqui-linear form on the pair of hyperboloids

Let us consider the following sesqui-linear form $\mathcal{A}(h,f)$ defined on the pair $(\mathcal{D}(\mathcal{X}),$ $\mathcal{D}(\mathcal{Y}))$:

$$\mathcal{A}(h,f) = \int_{\mathcal{X}\times\mathcal{Y}} \delta([x,y])h(x)\overline{f(y)}dxdy.$$

The main result of our work consists of Theorem 6.1, which gives the decomposition of this form in terms of Fourier components of functions h and f. The decomposition contains Fourier components of the continuous series ($\sigma = -1/2 + i\rho$).

Theorem 6.1. *The sesqui-linear form $\mathcal{A}(h,f)$ decomposes into Fourier components of the continuous series $F_{\sigma,0}f$ and $G_\sigma h$, $\sigma = -1/2 + i\rho$, $\rho \in \mathbb{R}$, as follows:*

$$\mathcal{A}(h,f) = \int_{-\infty}^{\infty} \mu(\sigma)\langle F_{\sigma,0}h, G_\sigma f\rangle_S \Big|_{\sigma=-1/2+i\rho} d\rho, \tag{6.1}$$

where

$$\mu(\sigma) = 2\omega(\sigma)\mathrm{B}\left(-\frac{\sigma}{2}, \frac{1}{2}\right) \tag{6.2}$$

$$= \frac{1}{16\pi^2}(2\sigma+1)\cot\sigma\pi \cdot \mathrm{B}\left(-\frac{\sigma}{2}, \frac{1}{2}\right), \tag{6.3}$$

the factor $\omega(\sigma)$ is given by (5.2), so that

$$\mu\left(-\frac{1}{2}+i\rho\right) = \frac{1}{8}\pi^{-5/2}\rho\tanh\rho\pi \cdot \sin\left(\frac{1}{4}+\frac{i\rho}{2}\right)\pi \cdot \left|\Gamma\left(\frac{1}{4}+\frac{i\rho}{2}\right)\right|^2$$

P r o o f. Let us take in (5.1) as φ the characteristic function of the interval $[0,a]$ divided by a and as ψ the function Mf, $f \in \mathcal{D}(\mathcal{Y})$. We can consider that $a \in [0,1]$. We obtain

$$\frac{1}{a}\int_0^a \overline{Mf(c)}\,dc = \sum_\varepsilon \int_{-\infty}^{\infty} \Omega_\varepsilon(\rho)\Big[\,\frac{1}{a}\int_0^a \overline{\Phi_{-1/2+i\rho,\varepsilon}(c)}dc\,\Big]\,d\rho, \tag{6.4}$$

where we denoted

$$\Omega_\varepsilon(\rho) = \omega(\sigma)\,(\Phi_{\sigma,\varepsilon}, Mf)\,\Big|_{\sigma=-1/2+i\rho}$$

$$= \langle \Psi_{\sigma,\varepsilon}, f\rangle_{\mathcal{Y}}\Big|_{\sigma=-1/2+i\rho},$$

see (4.7). Let a tend to 0. Then the left hand side of (6.4) goes to $\overline{Mf(0)}$. Let us prove that we can pass to the limit under the integral over ρ in the right hand side of (6.4). By the mean value theorem, the integral in the right hand side of (6.4) is equal to

$$F_\varepsilon(a) = \int_{-\infty}^{\infty} \Omega_\varepsilon(\rho)\overline{\Phi_{-1/2+i\rho,\varepsilon}(\eta)}d\rho, \tag{6.5}$$

where η is a number in $[0,a]$ (depending on a, ρ and ε). We have to prove that

$$F_\varepsilon(a) \to F_\varepsilon(0) \tag{6.6}$$

when $a \to 0$, where

$$F_\varepsilon(0) = \int_{-\infty}^{\infty} \Omega_\varepsilon(\rho)\overline{\Phi_{-1/2+i\rho,\varepsilon}(0)}d\rho. \tag{6.7}$$

Let us take an arbitrary number $\gamma > 0$. In virtue of Theorem 4.2 both functions $\Omega_\varepsilon(\rho)$, $\varepsilon = 0, 1$, decrease rapidly when $|\rho| \to 0$. Therefore, there exists a number A such that

$$\int_{|\rho|\geqslant A} \Big|\Omega_\varepsilon(\rho)\Big| d\rho < \gamma. \tag{6.8}$$

It follows from formula (4.4) that the function $\Phi_{-1/2+i\rho,\varepsilon}(c)$ is bounded, i. e.

$$\Big|\Phi_{-1/2+i\rho,\varepsilon}(c)\Big| \leqslant N, \tag{6.9}$$

N being some number, for all $\rho \in \mathbb{R}$, $\varepsilon = 0, 1$, and all c from some finite interval, for example, $[0, 1]$. Indeed, formula (4.4) implies the inequality

$$\Big|\Phi_{\sigma,\varepsilon}(c)\Big| \leqslant \int_0^{2\pi} \Big|c + \sqrt{c^2+1}\cdot\cos\alpha\Big|^{-1/2} d\alpha \tag{6.10}$$

since the function of c in the right hand side of (6.10) (it is the function $\Phi_{-1/2,0}$) is continuous with respect to c.

On the other hand, since the function $\Phi_{-1/2+i\rho,\varepsilon}(c)$ is continuous with respect to ρ and c, there exists a number $\delta > 0$ such that

$$\Big|\Phi_{-1/2+i\rho,\varepsilon}(\eta) - \Phi_{-1/2+i\rho,\varepsilon}(0)\Big| < \gamma \tag{6.11}$$

for $|\rho| \leqslant A$ and $0 \leqslant \eta < \delta$. Then for $0 < a < \delta$ we have

$$\begin{aligned} \Big|F_\varepsilon(a) - F_\varepsilon(0)\Big| &\leqslant \int_{-\infty}^{\infty} \Big|\Omega_\varepsilon(\rho)\Big| \cdot \Big|\Phi_{-1/2+i\rho,\varepsilon}(\eta) - \Phi_{-1/2+i\rho,\varepsilon}(0)\Big| d\rho \\ &= \int_{-A}^{A} + \int_{|\rho|\geqslant A} \\ &\leqslant \gamma\int_{-A}^{A} \Big|\Omega_\varepsilon(\rho)\Big| d\rho + 2N\int_{|\rho|\geqslant A} \Big|\Omega_\varepsilon(\rho)\Big| d\rho \\ &\leqslant (C_\varepsilon + 2N)\gamma, \end{aligned} \tag{6.12}$$

where

$$C_\varepsilon \int_{-\infty}^{\infty} \Big|\Omega_\varepsilon(\rho)\Big| d\rho,$$

here we used (6.5), (6.7)–(6.9), (6.11). Inequality (6.12) proves (6.6).

Now we may pass to the limit in (6.4) when $a \to 0$. We obtain

$$\overline{Mf(0)} = \int_{-\infty}^{\infty} \omega(\sigma) \sum_\varepsilon \overline{\Phi_{\sigma,\varepsilon}(0)}\ \langle \Psi_{\sigma,\varepsilon}, f\rangle_{\mathcal{Y}}\Big|_{\sigma=-1/2+i\rho} d\rho. \tag{6.13}$$

By (4.4) we have

$$\begin{aligned} \Phi_{\sigma,\varepsilon}(0) &= \int_0^{2\pi} (\cos\varphi)^{\sigma,\varepsilon} d\varphi \\ &= [\,1 + (-1)^\varepsilon\,]\,\mathrm{B}\left(\frac{\sigma+1}{2},\ \frac{1}{2}\right). \end{aligned}$$

We see that $\Phi_{\sigma,\varepsilon}(0)$ is equal to zero for $\varepsilon = 1$, so that only one summand in (6.13) remains – with $\varepsilon = 0$. Since

$$\overline{\sigma} = -\sigma - 1 \quad \text{for} \quad \sigma = -1/2 + i\rho, \tag{6.14}$$

equality (6.13) is

$$\overline{Mf(0)} = \int_{-\infty}^{\infty} \mu(\sigma) \langle \Psi_{\sigma,0}, f\rangle_{\mathcal{Y}} \Big|_{\sigma=-1/2+i\rho} d\rho, \tag{6.15}$$

where $\mu(\sigma)$ is given by (6.3), (6.4).

The left hand side in (6.15) is

$$\overline{Mf(0)} = \int_{\mathcal{Y}} \delta\left([x^0, y]\right) \overline{f(y)}\, dy. \tag{6.16}$$

Taking into account (6.16) let us apply (6.15) to a shifted function $\left(U_{\mathcal{Y}}(g)f\right)(y) = f(yg)$, $g \in G$. We get

$$\int_{\mathcal{Y}} \delta\left([x, y]\right) \overline{f(y)} dy = \int_{-\infty}^{\infty} \mu(\sigma) \langle \Psi_{\sigma,0}, U_{\mathcal{Y}}(g) f\rangle_{\mathcal{Y}} \Big|_{\sigma=-1/2+i\rho} d\rho, \tag{6.17}$$

where $x = x^0 g$.

Now multiply both sides of (6.17) by a function $h(x)$ in $\mathcal{D}(\mathcal{X})$ and integrate over $x \in \mathcal{X}$. In the right hand side we may invert the order of integrations – in virtue of Lemma 6.1, see below. We obtain:

$$\mathcal{A}(h, f) = \int_{-\infty}^{\infty} \mu(\sigma) \int_{\mathcal{X}} \langle \Psi_{\sigma,0}, U_{\mathcal{Y}}(g) f\rangle_{\mathcal{Y}}\, h(x) dx \Big|_{\sigma=-1/2+i\rho} d\rho.$$

The integral over $\mathcal{X}$ is nothing but the functional $K(\Psi_{\sigma,0}|h, f)$. Substituting its expression (4.10) in terms of Fourier components and taking into account (6.14), we get (6.1). □

Lemma 6.1. *For any function $f(y)$ in $\mathcal{D}_{\mathcal{Y}}$ the integral in the right hand side of (6.17) converges absolutely and uniformly with respect to $x = x^0 g$ on any compact $V \subset \mathcal{X}$.*

P r o o f. The hyperboloid $\mathcal{X}$ can be embedded into the group G as the product AK of subgroups A and K. By continuity of $U_{\mathcal{Y}}$, the union of supports of all functions $U_{\mathcal{Y}}(g)f$, where $g = ak$ is such that $x^0 g \in V$ is some compact W in $\mathcal{Y}$. By Theorem 4.1 there exists $C > 0$ such that for any $g = ak$, $x^0 g \in V$, the following inequality holds

$$\left|\langle \Psi_{\sigma,\varepsilon}, U_{\mathcal{Y}}(g) f\rangle_{\mathcal{Y}}\right| \leqslant C \cdot \max_y \left|\left(\Delta_{\mathcal{Y}}^m U_{\mathcal{Y}}(g) f\right)(y)\right| \cdot (\rho^2 + 1/4)^{-m}.$$

Since $\Delta_{\mathcal{Y}}$ commutes with translations, we have

$$\max_y \left| \left(\Delta_{\mathcal{Y}}^m U_{\mathcal{Y}}(g) f\right)(y)\right| = \max_y \left|\left(U_{\mathcal{Y}}(g) \Delta_{\mathcal{Y}}^m f\right)(y)\right| = \max_y \left|\Delta_{\mathcal{Y}}^m f(y)\right|,$$

so that there exist numbers C_m, $m \in \mathbb{N}$, such that

$$\left|\langle \Psi_{\sigma,\varepsilon}, U_{\mathcal{Y}}(g) f\rangle_{\mathcal{Y}}\right| \leqslant C_m \cdot (\rho^2 + 1/4)^{-m},$$

for all $x = x^0 g \in V$ and all $m \in \mathbb{N}$, whence the lemma. □

The quasiregular representation of $G = \mathrm{SO}_0(1,2)$ on $\mathcal{X}$ contains representations of the continuous series with multiplicity two and the analytic and antianalytic series with multiplicity one, and the quasiregular representation of G on $\mathcal{Y}$ contains representations of the continuous series with multiplicity one. Theorem 6.1 gives

Theorem 6.2. *The kernel of the Radon transform R consists of functions belonging to the discrete spectrum and to the odd part of the continuous spectrum on $\mathcal{X}$, its image goes in $C^\infty(\mathcal{Y})$. The kernel of the Radon transform R^* is $\{0\}$, its image consists of functions belonging to the even part of the continuous spectrum $\mathcal{X}$.*

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